

Functional-differential equations for the q -Fourier transform of q -Gaussians

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Abstract

In the paper the question - *Is the q -Fourier transform of a q -Gaussian a q' -Gaussian (with some q') up to a constant factor?* - is studied for the whole range of $q \in (-\infty, 3)$. This question is connected with applicability of the q -Fourier transform in the study of limit processes in nonextensive statistical mechanics. We prove that the answer is affirmative if and only if $q \geq 1$, excluding two particular cases of $q < 1$, namely, $q = \frac{1}{2}$ and $q = \frac{2}{3}$, which are also out of the theory valid for $q \geq 1$. We also discuss some applications of the q -Fourier transform to nonlinear partial differential equations such as the porous medium equation.

1 Introduction

Approximately one century after Boltzmann's seminal works which have turn into the cornerstones of statistical mechanics, Tsallis [1] introduced an entropic form aimed to accommodate the description of systems whose fundamental features do not fit for the properties assumed in the Boltzmann-Bibbs formalism; see [2, 3, 4]. Tsallis' entropic form, which is usually called non-additive q -entropy, recovers the classic Boltzmann-Gibbs entropic form in limit the case $q \rightarrow 1$. Concomitantly, there is the nonextensive statistical mechanics formalism based on the q -algebra and the q -Gaussian probability density function, which maximises q -entropy under certain appropriate constraints (see [4, 5, 6] and references therein). Recently, the q -Fourier transform was introduced [7] as a tool for the study of attractors of strongly correlated random variables arising in nonextensive statistical mechanics. In this paper, we shed light on the question - whether the q -Fourier transform

of a q -Gaussian is a q' -Gaussian for some another q' again. A key to this question is crucial because, as a mathematical tool, the F_q is relevant to both the study of limit distributions and, as we will show later on this paper, the solution of partial differential equations with physical significance as well. Moreover, a positive answer implies validating the mapping relation of q onto q' obtained from F_q , which has been predominant for the establishment of other stable distributions, namely the (q, α) -stable distributions [11]. We recall that, by definition, the F_q -transform, or q -Fourier transform of a nonnegative $f \in L_1(R)$ is defined by the formula

$$F_q[f](\xi) = \int_{\text{supp } f} e_q^{ix\xi} \otimes_q f(x) dx, \quad (1)$$

where $q < 3$, the symbol \otimes_q stands for the q -product, and

$$e_q^z = (1 + (1 - q)z)^{1/(1-q)}, \quad z \in C, \quad (2)$$

is a q -exponential (see [6, 7] for details). The equality

$$e_q^{ix\xi} \otimes_q f(x) = f(x) e_q^{\frac{ix\xi}{[f(x)]^{1-q}}},$$

valid for all $x \in \text{supp } f$, implies the following representation for the q -Fourier transform without usage of the q -product:

$$F_q[f](\xi) = \int_{\text{supp } f} f(x) e_q^{ix\xi [f(x)]^{q-1}} dx. \quad (3)$$

The remaining of the paper is organised as follows: In Section 2 we mention some properties of F_q ; In Section 3 we derive functional-differential equations for the q -Fourier transform of q -Gaussians. Then, based on the results of this Section, we show that the answer to the above question is affirmative for all $q \geq 1$, and for two particular values of $q < 1$, namely for $q = 1/2$ and $q = 2/3$. We also show that if $q < 1$, except two values mentioned above, F_q -transform of a q -Gaussian is no longer a q' -Gaussian, $\forall q' < 3$. A relevant physical application of F_q and the functional-differential equations studied in Section 3 is addressed in Section 4.

2 Preliminaries

The following properties of F_q follow immediately from its representation (3).

Proposition 2.1 *For any constants $a > 0$, $b > 0$,*

1. $F_q[af(x)](\xi) = aF_q[f(x)](\frac{\xi}{a^{1-q}})$;
2. $F_q[f(bx)](\xi) = \frac{1}{b}F_q[f(x)](\frac{\xi}{b})$.

Now we recall some facts related to q -Gaussians. Let β be a positive number. A function

$$G_q(\beta; x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}, \quad (4)$$

is called a q -Gaussian. The constant C_q is the normalising constant, namely $C_q = \int_{-\infty}^{\infty} e_q^{-x^2} dx$, with explicit expression [7]

$$C_q = \begin{cases} \frac{2}{\sqrt{1-q}} \int_0^{\pi/2} (\cos t)^{\frac{3-q}{1-q}} dt = \frac{2\sqrt{\pi} \Gamma(\frac{1}{1-q})}{(3-q)\sqrt{1-q} \Gamma(\frac{3-q}{2(1-q)})}, & -\infty < q < 1, \\ \sqrt{\pi}, & q = 1, \\ \frac{2}{\sqrt{q-1}} \int_0^{\infty} (1+y^2)^{\frac{-1}{q-1}} dy = \frac{\sqrt{\pi} \Gamma(\frac{3-q}{2(q-1)})}{\sqrt{q-1} \Gamma(\frac{1}{q-1})}, & 1 < q < 3. \end{cases} \quad (5)$$

If $q < 1$, then $G_q(\beta; x)$ has a compact support $|x| \leq K_\beta$, where $K_\beta = (\beta(1-q))^{-1/2}$. We use the convention $K_\beta = \infty$ if $q \geq 1$, since the support of a q -Gaussian is not bounded in this case.

Note that q -exponentials possess the property $e_q^z \otimes_q e_q^w = e_q^{z+w}$ [12, 13]. This implies the following proposition.

Proposition 2.2 *For all $q < 3$ the q -Fourier transform of $e_q^{-\beta x^2}$, $\beta > 0$, can be written in the form*

$$F_q[e_q^{-\beta|x|^2}](\xi) = \int_{-K_\beta}^{K_\beta} e_q^{-\beta|x|^2 + ix\xi} dx. \quad (6)$$

Corollary 2.3 *Let $q < 3$. Then*

$$F_q[e_q^{-\beta|x|^2}](\xi) = 2 \int_0^{K_\beta} e_q^{-\beta|x|^2} \cosh_q \left(\frac{x\xi}{[e_q^{-\beta|x|^2}]^{1-q}} \right) dx, \quad \forall q,$$

where

$$\cosh_q(x) = \frac{e_q^x + e_q^{-x}}{2}.$$

The following assertion was proved in [7].

Proposition 2.4 *Let $1 \leq q < 3$. Then*

$$F_q[G_q(\beta; x)](\xi) = e_{q_1}^{-\beta_* \xi^2}, \quad (7)$$

where $q_1 = \frac{1+q}{3-q}$ and $\beta_* = \frac{3-q}{8\beta^{2-q}C_q^{2(q-1)}}$.

Proposition 2.5 *Let $q < 1$. Then*

$$F_q[G_q(\beta, x)] = e_{q_1}^{-\beta_* |\xi|^2} \left(1 - \frac{2}{C_q} \operatorname{Im} \int_0^{d_\xi} e_q^{b_\xi + i\tau} d\tau\right),$$

where $q_1 = (1+q)/(3-q)$, C_q is the normalising constant and $b_\xi + id_\xi = \frac{K_\beta \sqrt{\beta} - i \frac{\xi}{2\sqrt{\beta}}}{[e_q^{-\frac{\xi^2}{4\beta}}]^{\frac{1-q}{2}}}$.

Proof. The proof of this statement can be obtained applying the Cauchy theorem, that is by integrating the function $e_q^{-\beta z^2 + iz\xi}$ over the closed contour $C = C_0 \cup C_1 \cup C_- \cup C_+$, where $C_p = (-K_\beta + pi, K_\beta + ip)$, $p = 0, 1$, and $C_\pm = [\pm K_\beta, \pm K_\beta + i]$. ■

It follows from Propositions 2.4 and 2.5 that

$$F_q[G_q(\beta, x)] = e_{q_1}^{-\beta_* |\xi|^2} + I_{(q<1)}(q) T_q(\xi),$$

where $I_{(a,b)}(\cdot)$ is the indicator function of (a, b) , and

$$T_q(\xi) = -\frac{2}{C_q} e_{q_1}^{-\beta_* |\xi|^2} \operatorname{Im} \int_0^{d_\xi} e_q^{b_\xi + i\tau} d\tau.$$

Thus for $q \geq 1$ F_q transforms a q -Gaussian into a q_1 -Gaussian with the factor $C_{q_1} \beta^{-1/2}$. However, for $q < 1$, the tail $T_q(\xi)$ appears.

Proposition 2.6 *For any real $q_1, \beta_1 > 0$ and $\delta > 0$ there exist uniquely determined $q_2 = q_2(q_1, \delta)$ and $\beta_2 = \beta_2(\delta, \beta_1)$, such that*

$$(e_{q_1}^{-\beta_1 x^2})^\delta = e_{q_2}^{-\beta_2 x^2}.$$

Moreover, $q_2 = \delta^{-1}(\delta - 1 + q_1)$, $\beta_2 = \delta\beta_1$.

Proof. Let $q_1 < 3, \beta_1 > 0$, and $\delta > 0$ be any fixed real numbers. For the equation,

$$(1 - (1 - q_1)\beta_1 x^2)^{\frac{\delta}{1-q_1}} = (1 - (1 - q_2)\beta_2 x^2)^{\frac{1}{1-q_2}}$$

to be an identity, it is needed $(1 - q_1)\beta_1 = (1 - q_2)\beta_2$, $1 - q_1 = \delta(1 - q_2)$. These equations have a unique solution $q_2 = \delta^{-1}(\delta - 1 + q_1)$, $\beta_2 = \delta\beta_1$. ■

Corollary 2.7 $(e_q^{-\beta x^2})^q = e_{2-\frac{1}{q}}^{-q\beta x^2}$.

Now we introduce a sequence q_n defined by the relation

$$q_n = \frac{2q - n(q-1)}{2 - n(q-1)}, \quad (8)$$

where $-\infty < n < \frac{2}{q-1} - 1$ if $1 < q < 3$, and $n > -\frac{2}{1-q}$ if $q \leq 1$. Notice that $q_n = 1$ for all $n = 0, \pm 1, \dots$, if $q = 1$. Let \mathbb{Z} be the set of all integer numbers. Denote by \mathbb{N}_q a subset of \mathbb{Z} defined as

$$\mathbb{N}_q = \begin{cases} \{n \in \mathbb{Z} : n < \frac{2}{q-1} - 1\}, & \text{if } 1 < q < 3, \\ \{n \in \mathbb{Z} : n > -\frac{2}{1-q}\}, & \text{if } q \leq 1. \end{cases}$$

Proposition 2.8 For all $n \in \mathbb{N}_q$ the relations

1. $(3 - q_n)q_{n+1} = (3 - q_{n-2})q_n$,
2. $2C_{q_{n-2}} = \sqrt{q_n}(3 - q_n)C_{q_n}$

hold true.

Proof. 1. It follows from the definition of q_n that $q_{n+1} = (1 + q_n)/(3 - q_n)$. This yields

$$(3 - q_n)q_{n+1} = 1 + q_n = (1 + \frac{1}{q_n})q_n. \quad (9)$$

Further, the duality relation $q_{k-1} + q_{k+1}^{-1} = 2$ holds for all $k \in \mathbb{N}_q$. Applying it for $k = n - 1$, we have $1/q_n = 2 - q_{n-2}$. Taking this into account in (9) we arrive at statement 1).

2. For $q = 1$, the relation 2) is reduced to the simple equality $2\sqrt{\pi} = 2\sqrt{\pi}$. Let $q \neq 1$. Notice that if $1 < q < 3$ then, $1 < q_n < 3$ for all $n \in \mathbb{N}_q$; if $q < 1$ then, $q_n < 1$ as well for all $n \in \mathbb{N}_q$. Consider $A_n = 2C_{n-2}/C_n$. Using the explicit form of C_q given in (5) and the duality relation $2 - q_{n-2} = 1/q_n$, in the case $1 < q < 3$ one obtains

$$A_n = \frac{\sqrt{q_n} \Gamma\left(\frac{1+q_n}{2(q_n-1)}\right)}{\frac{1}{2(q_n-1)} \Gamma\left(\frac{3-q_n}{2(q_n-1)}\right)} = \sqrt{q_n}(3 - q_n).$$

Further, if $q < 1$, then

$$A_n = \frac{\sqrt{q_n}(3 - q_n)}{\frac{1+q_n}{2(1-q_n)} \Gamma\left(\frac{1+q_n}{2(1-q_n)}\right)} = \sqrt{q_n}(3 - q_n),$$

proving the statement 2). ■

3 Main results

3.1 Functional differential equations

Denote $g_q(\beta, \xi) = F_q[G_q(\beta, x)](\xi)$. For $\beta = 1$, we use the notation $g_q(\xi) = g_q(1, \xi)$. Let $Y(q, \xi) = F_q[e_q^{-x^2}](\xi)$. By Proposition 2.2,

$$Y(q, \xi) = \int_{-K}^K e_q^{-|x|^2+ix\xi} dx,$$

where $K = K_1 = \frac{1}{\sqrt{1-q}}$ if $q < 1$, and $K = \infty$, if $q \geq 1$.

Lemma 3.1 *For any $q < 3$ and $\beta > 0$ we have,*

1. $g_q(\beta, \xi) = g_q(\frac{\xi}{(\sqrt{\beta})^{2-q}})$;
2. $g_q(\xi) = \frac{1}{C_q} Y(q, C_q^{1-q} \xi)$.

Proof. The proof follows from the properties of F_q indicated in Proposition 2.1.

These two formulae imply,

$$F_q[G_q(\beta, x)](\xi) = \frac{1}{C_q} Y\left(q, \left(\frac{C_q}{\sqrt{\beta}}\right)^{1-q} \frac{\xi}{\sqrt{\beta}}\right).$$

Moreover, $g_q(\beta, 0) = 1$, which implies $g_q(0) = 1$ and $Y(q, 0) = C_q$. Thus, it suffices to study $Y(q, \xi)$ in order to know properties of the q -Fourier transform of q -Gaussians.

Theorem 3.2 *Let $1 \leq q < 3$ and $q_n, n \in \mathbb{N}_q$, are defined in (8). Then $Y(q_n, \xi)$ satisfies the following homogeneous functional-differential equation*

$$2\sqrt{q_n} \frac{\partial Y(q_n, \xi)}{\partial \xi} + \xi Y(q_{n-2}, \sqrt{q_n} \xi) = 0; \quad (10)$$

Proof. Differentiating $Y(q, \xi) = \int_{-K}^K e_q^{-x^2+ix\xi}$ with respect to ξ , we have

$$\frac{\partial Y(q, \xi)}{\partial \xi} = i \int_{-K}^K x (e_q^{-x^2+ix\xi})^q dx.$$

Further, integrating by parts, we obtain

$$\frac{\partial Y(q, \xi)}{\partial \xi} = \frac{-i}{2} \int_{-K}^K d(e_q^{-x^2+ix\xi}) - \frac{\xi}{2} \int_{-K}^K (e_q^{-x^2+ix\xi})^q dx. \quad (11)$$

It is not straightforward to see that the first integral vanishes if $q \geq 1$. Applying Corollary 2.7, the second integral can be represented in the form

$$\int_{-K}^K (e_q^{-x^2+ix\xi})^q dx = \frac{1}{\sqrt{q}} \int_{-K}^K e_{2^{-1/q}}^{-x^2+ix\sqrt{q}\xi} dx = \frac{1}{\sqrt{q}} Y\left(2 - \frac{1}{q}, \sqrt{q}\xi\right). \quad (12)$$

Hence, for $q \geq 1$ the function $F_q[e_q^{-x^2}]$ satisfies the functional-differential equation

$$2\sqrt{q} \frac{\partial Y(q, \xi)}{\partial \xi} + \xi Y(2 - 1/q, \sqrt{q}\xi) = 0. \quad (13)$$

Now let $q = q_n, n \in \mathbb{N}_q$. Then taking into account the relation $2 - 1/q_n = q_{n-2}$ we obtain (10). ■

Theorem 3.3 *Let $0 < q < 1$ and $q \neq l/(l+1), l = 1, 2, \dots$. Then $Y(q_n, \xi)$ satisfies the following inhomogeneous functional-differential equation*

$$2\sqrt{q_n} \frac{\partial Y(q_n, \xi)}{\partial \xi} + \xi Y(q_{n-2}, \sqrt{q_n}\xi) = r_{q_n} \xi^{\frac{1}{1-q_n}}, \quad (14)$$

where

$$r_{q_n} = 2\sqrt{q_n} \sin \frac{\pi}{2(1-q_n)} (1 - q_n)^{\frac{1}{2(1-q_n)}}. \quad (15)$$

Proof. Assume that $q < 1$ and $q \neq l/(l+1), l = 1, 2, \dots$. We notice that if $q < 1$ then the first integral on the right hand side of (11) does not vanish. Now it takes the form

$$\int_{-K}^K d(e_q^{-x^2+ix\xi}) = e_q^{-K^2+iK\xi} - e_q^{-K^2-iK\xi} = 2i \operatorname{Im} e_q^{-K^2+iK\xi}.$$

Since $\operatorname{supp} e_q^{-x^2} = [-K, K]$, one has $e_q^{-K^2} = 0$. Hence,

$$e_q^{-K^2+iK\xi} = 0 \otimes_q e_q^{iK\xi} = [i(1-q)K\xi]^{\frac{1}{1-q}}.$$

Further, taking into account $K = 1/\sqrt{1-q}$, we obtain

$$\operatorname{Im}[i(1-q)K\xi]^{\frac{1}{1-q}} = (1-q)^{\frac{1}{2(1-q)}} \sin \frac{\pi}{2(1-q)} \xi^{\frac{1}{1-q}}.$$

The expression in (12) for the second integral in the right hand side of (11) is the same in the case of $q < 1$. Hence, in this case $F_q[e_q^{-x^2}](\xi)$ satisfies the functional-differential equation

$$2\sqrt{q} \frac{\partial Y(q, \xi)}{\partial \xi} + \xi Y(2 - 1/q, \sqrt{q}\xi) = r_q \xi^{\frac{1}{1-q}}, \quad (16)$$

where

$$r_q = 2\sqrt{q}(1-q)^{\frac{1}{2(1-q)}} \sin \frac{\pi}{2(1-q)}.$$

Again, by taking $q = q_n, n \in \mathbb{N}_q$, we arrive at the functional-differential equation (14). ■

Now we consider the case $q = l/(l+1)$, $l = 1, 2, \dots$, excluded from Theorems 3.2 and 3.3. In this case $K = \sqrt{l+1}$ and $Y(q, \xi)$ takes the form

$$Y(q, \xi) = F_q[e_q^{-x^2}](\xi) = \int_{-\sqrt{l+1}}^{\sqrt{l+1}} (1 - \frac{1}{l+1}x^2 + \frac{1}{l+1}ix\xi)^{l+1} dx.$$

We use notation $P_{l+1}(\xi) = Y(\frac{l}{l+1}, \xi)$ indicating the dependence on l . Further, obviously

$$2 - \frac{1}{q} = \frac{l-1}{l},$$

and consequently

$$Y(2 - 1/q, \xi) = \int_{-\sqrt{l}}^{\sqrt{l}} (1 - \frac{1}{l}x^2 + \frac{1}{l}ix\xi)^l dx = P_l(\xi).$$

It is simple to see that $P_l(\xi)$ is a polynomial of even order, namely of order l if l is even, and of order $l-1$ if l is odd. Moreover, $P_l(\xi)$ is a symmetric function of ξ and $P_l(0) = C_{\frac{l-1}{l}} > 0$. Let ρ be a root of $P_l(\xi)$ closest to the origin. We will consider $P_l(\xi)$ only on the interval $\xi \in [-\rho, \rho]$, where it is positive.

Theorem 3.4 *Let $q = \frac{2m-1}{2m}$, $m = 1, 2, \dots$. Then $Y(q, \xi)$ satisfies the functional-differential equation (10).*

Proof. Assume $l+1 = 2m, m = 1, 2, \dots$. In this case $Y(q, \xi) = P_{2m}(\xi)$ is a polynomial of order $2m$ and $Y(2 - 1/q, \xi) = P_{2m-1}(\xi)$ is a polynomial of order $2m-1$. Moreover, it is easy to check that in this case $r_q = 0$. Thus, $Y(q, \xi)$ satisfies the equation

$$2\sqrt{q} \frac{\partial Y(q, \xi)}{\partial \xi} + \xi Y(2 - 1/q, \sqrt{q}\xi)(\xi) = 0. \quad (17)$$

It is easy to verify that this equation is consistent. ■

Theorem 3.5 *Let $q = \frac{2m}{2m+1}$, $m = 1, 2, \dots$. Then $Y(q, \xi)$ satisfies neither the functional-differential equation (10) nor (14).*

Proof. Let $l = 2m, m = 1, 2, \dots$. Then $Y(q, \xi) = P_{2m+1}(\xi)$ is a polynomial of order $2m$, as well as $Y(2 - 1/q, \xi) = P_{2m}(\xi)$. Assume $Y(q, \xi)$ satisfies the equation (14), which in this case takes the form

$$2\sqrt{q}\frac{\partial Y(q, \xi)}{\partial \xi} + \xi P_{2m}(\xi) = \frac{(-1)^m}{(2m-1)^{m-\frac{1}{2}}}\xi^{2m+1}. \quad (18)$$

Clearly the derivative of a polynomial of order $2m$ can not be a polynomial of order $2m+1$. Analogously, $Y(q, \xi)$ cannot satisfy equation (10) either. ■

3.2 Is the q -Fourier transform of a q -Gaussian a q' -Gaussian?

We shall now introduce the set of functions

$$\mathcal{G} = \bigcup_{q < 3} \mathcal{G}_q, \text{ where } \mathcal{G}_q = \{f : f(x) = ae_q^{-\beta x^2}, a > 0, \beta > 0\}. \quad (19)$$

Theorem 3.6 *Let $1 \leq q_n < 3$. Then the following Cauchy problem for a functional-differential equation*

$$2\sqrt{q_n}\frac{\partial Y(q_n, \xi)}{\partial \xi} + \xi Y(q_{n-2}, \sqrt{q_n}\xi) = 0; \quad (20)$$

$$Y(q_n, 0) = C_{q_n}, \quad (21)$$

has a solution $Y(q_n, \xi) \in \mathcal{G}$ and this solution is specifically

$$Y(q_n, \xi) = C_{q_n} e_{q_{n+1}}^{-\frac{3-q_n}{8}\xi^2}. \quad (22)$$

Proof. It immediately follows from the representation that $Y(q_n, 0) = C_{q_n}$. Furthermore,

$$\begin{aligned} \frac{\partial Y(q_n, \xi)}{\partial \xi} &= -\frac{1}{4}(3 - q_n) C_{q_n} \xi \left(e_{q_{n+1}}^{-\frac{3-q_n}{8}\xi^2} \right)^{q_{n+1}}, \\ Y(q_{n-2}, \sqrt{q_n}\xi) &= C_{q_{n-2}} e_{q_{n-1}}^{-q_n \frac{3-q_{n-2}}{8}\xi^2}. \end{aligned} \quad (23)$$

In addition, Corollary 2.7 and relation 1) in Proposition 2.6 imply that

$$\frac{\partial Y(q_n, \xi)}{\partial \xi} = -\frac{1}{4}(3 - q_n) C_{q_n} \xi e_{q_{n-1}}^{-q_n \frac{3-q_{n-2}}{8}\xi^2}. \quad (24)$$

Substituting (23) and (24) in equation (20), we obtain

$$(-\sqrt{q_n} C_{q_n} \frac{3 - q_n}{2} + C_{q_{n-2}}) e_{q_{n-1}}^{-\frac{q_n(3-q_n)}{8}\xi^2} = 0. \quad (25)$$

Now taking into account the second relation in Proposition 2.6 we conclude that $Y(q_n, \xi)$ in (22) satisfies equation (20). ■

Corollary 3.7 *Let $q_n \geq 1$. Then*

$$F_{q_n}[G_{q_n}](\xi) = e_{q_n+1}^{-\frac{3-q_n}{8\beta^{2-q_n}C_{q_n}^{2(q_n-1)}}\xi^2}.$$

Remark 3.8 *Representation (26) was obtained in [7] by the contour integration technique.*

Theorem 3.9 *Let $q_n < 1$, $n \in \mathbb{N}$ and $q_n \neq m/(m+1)$, $m = 1, 2, \dots$. Then the Cauchy problem for a functional-differential equation*

$$2\sqrt{q_n}\frac{\partial Y(q_n, \xi)}{\partial \xi} + \xi Y(q_{n-2}, \sqrt{q_n}\xi) = r_{q_n}\xi^{\frac{1}{1-q_n}}, \quad (26)$$

$$Y(q_n, 0) = C_{q_n}, \quad (27)$$

has no solution in \mathcal{G} .

Proof. Let $q_n < 1$, $q_n \neq m/(m+1)$, $m = 1, 2, \dots$. First, we notice that a function with compact support can not solve equation (26). It follows that a solution to (26), $Y(q_n, \xi) \notin \mathcal{G}_q$ with $q < 1$, since any function in \mathcal{G}_q for $q < 1$ has compact support. Now assume that there exists a $q = q(q_n) \geq 1$, such that $Y(q_n, \xi) \in \mathcal{G}_q$, that is

$$Y(q_n, \xi) = A_{q_n}e_q^{-b(q_n)\xi^2},$$

where $A_{q_n} > 0$, $b(q_n) > 0$ are some real numbers. It follows from Eq. (27) that $A_{q_n} = C_{q_n}$. Further, $Y(q_{n-2}, \xi) \in \mathcal{G}_{q^*}$, where $q^* = 2 - 1/q$, that is $Y(q_{n-2}, \xi) = C_{q_{n-2}}e_{q^*}^{-\beta(q_n)\xi^2}$, $\beta(q_n) > 0$. Then, for $Y(q_n, \xi)$ to be consistent with Eq. (26), one has

$$\frac{2}{1-q^*} + 1 = \frac{1}{1-q_n},$$

or $q^* = \frac{3q_n-2}{q_n}$. Hence, $q = \frac{q_n}{2-q_n} < 1$, since $q_n < 1$. This contradicts the assumption that $q \geq 1$. ■

Finally, considering the specific cases $q = \frac{1}{2}, \frac{2}{3}, \dots, \frac{m}{m+1}, \dots$ a direct computation shows that

$$F_{\frac{1}{2}}\left[e_{\frac{1}{2}}^{-x^2}\right](\xi) = \frac{16\sqrt{2}}{15}\left(1 - \frac{5}{16}\xi^2\right).$$

This function is non-negative for $|\xi| \leq 4/\sqrt{5}$, so in this interval we can associate it by $\frac{16\sqrt{2}}{15}e_0^{-(5/16)\xi^2} \in \mathcal{G}_0$. The similar situation holds true in the case $q = 2/3$ as well yielding,

$$F_{\frac{2}{3}}\left[e_{\frac{2}{3}}^{-x^2}\right](\xi) = \frac{32\sqrt{3}}{35}\left(1 - \frac{7}{24}\xi^2\right),$$

which is positive in the interval $(-\frac{2\sqrt{6}}{7}, \frac{2\sqrt{6}}{7})$.

Below we show that for all values of $q = 3/4, 4/5, \dots$ F_q -transform of $e_q^{-x^2}$ does not belong to \mathcal{G} . First we obtain an explicit form for $P_{m+1}(\xi) = F_q[e_q^{-x^2}]$. Recall that $P_{m+1}(\xi)$ is a polynomial of order $m+1$ if $m+1$ is even. Otherwise it is a polynomial of order m .

Theorem 3.10 *Let $q = m/(m+1), m = 1, 2, \dots$. Then $Y(q, \xi) = P_{m+1}(\xi)$ is represented in the form*

$$P_{m+1}(\xi) = \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} (-1)^k \binom{m+1}{2k} (m+1)^{-k+\frac{1}{2}} B\left(k + \frac{1}{2}, m - 2k + 2\right) \xi^{2k}, \quad (28)$$

where $[x]$ means the integer part of x , and $B(a, b)$ is the Euler's beta-function.

Proof. Recall that if $q = \frac{m}{m+1}, m = 1, 2, \dots$, then $Y(q, \xi)$ has the form

$$Y(q, \xi) = P_{m+1}(\xi) = \int_{-\sqrt{m+1}}^{\sqrt{m+1}} \left(1 - \frac{1}{m+1}x^2 + \frac{1}{m+1}ix\xi\right)^{m+1} dx.$$

We have

$$P_{m+1}(\xi) = \sum_{k=0}^{m+1} \binom{m+1}{k} D_k(m) \frac{(i\xi)^k}{(m+1)^k},$$

where

$$D_k(m) = \int_{-\sqrt{m+1}}^{\sqrt{m+1}} \left(1 - \frac{1}{m+1}x^2\right)^{m-k+1} x^k dx.$$

Explicitly $D_k(m) = 0$ if k is odd and $D_{2k}(m) = (m+1)^{k+1/2} B(k+1/2, m-2k+2)$ for $k = 0, \dots, \lfloor \frac{m+1}{2} \rfloor$ which leads to representation (28). ■

Theorem 3.11 *Let $q = m/(m+1), m = 3, 4, \dots$. Then $Y(q, \xi) \notin \mathcal{G}$.*

Proof. It follows from the representation (28) that the first three terms of the polynomial $Y(q, \xi)$ are

$$\begin{aligned} Y(q, \xi) &= P_{m+1}(\xi) \\ &= D_0(m) \left[1 - (m+1)^2 \frac{B(\frac{3}{2}, m)}{B(\frac{1}{2}, m+2)} \xi^2 + \frac{m(m+1)^3}{2} \frac{B(\frac{5}{2}, m-2)}{B(\frac{1}{2}, m+2)} \xi^4 + \dots \right] \\ &= D_0(m) \left[1 - \frac{2m+3}{8(m+1)} \xi^2 + \frac{(2m+3)(2m+1)}{8(m+1)^2} \xi^4 + \dots \right], \end{aligned} \quad (29)$$

where

$$D_0(m) = C \frac{m}{m+1} = \sqrt{m+1} B\left(\frac{1}{2}, m+2\right) = \frac{\sqrt{m+1}(m+1)!2^{m+2}}{(2m+3)!!}.$$

Now assume that $Y(q, \xi) \in \mathcal{G}_{q*}$ for some $q_* < 3$. Then $1/(1 - q_*) = (m+1)/2$, or $q_* = (m-1)/(m+1)$. We have

$$Y(q, \xi) = D_0(m)(1 - \beta(m)\xi^2)^{[\frac{m+1}{2}]},$$

where $\beta(m) > 0$ and $|\xi| \leq 1/\sqrt{\beta(m)}$. Applying the binomial formula and keeping the first three terms, one has

$$Y(q, \xi) = D_0(m) \left[1 - \frac{(m+1)\beta(m)}{2} \xi^2 + \frac{(m^2-1)[\beta(m)]^2}{8} \xi^4 + \dots \right]. \quad (30)$$

Comparing the second and third terms of (29) and (30), one obtains contradictory relations

$$\beta(m) = \frac{2m+3}{4(m+1)^2}$$

and

$$[\beta(m)]^2 = \frac{(3m+3)(2m+1)}{(m-1)(m+1)^3} \neq \frac{(2m+3)^2}{16(m+1)^4} = [\beta(m)]^2, \quad m = 3, 4, \dots$$

which proves the statement. ■

Remark 3.12 The formula (28) for $q = 1/2$ and $q = 2/3$ gives

$$F_{\frac{1}{2}} \left[e^{-\frac{1}{2}x^2} \right] (\xi) = \frac{16\sqrt{2}}{15} \left(1 - \frac{5}{16} \xi^2 \right) = \frac{16\sqrt{2}}{15} e_0^{-(5/16)\xi^2}, \quad \xi \in \left[-\frac{4\sqrt{5}}{5}, \frac{4\sqrt{5}}{5} \right],$$

and

$$F_{\frac{2}{3}} \left[e^{-\frac{2}{3}x^2} \right] (\xi) = \frac{32\sqrt{3}}{35} \left(1 - \frac{7}{24} \xi^2 \right) = \frac{32\sqrt{3}}{35} e_0^{-\frac{7}{24}\xi^2}, \quad \xi \in \left[-\frac{2\sqrt{6}}{7}, \frac{2\sqrt{6}}{7} \right].$$

Both functions belong to \mathcal{G}_0 .

Remark 3.13 If $q = 1$ then the Cauchy problem (10), (21) reads

$$2Y'(\xi) + \xi Y(\xi) = 0, \quad Y(0) = \sqrt{\pi},$$

and its unique solution is $Y(\xi) = \sqrt{\pi} e^{-\xi^2/4}$. Besides from Corollary 3.7 we obtain

$$F \left[\frac{\sqrt{\beta}}{\sqrt{\pi}} e^{-\beta x^2} \right] = e^{-\frac{1}{4\beta} \xi^2}.$$

The density of the standard normal distribution corresponds to $\beta = 1/2$, giving the characteristic function of the classic Gaussian.

4 Some applications to the porous medium equation

In this Section we discuss some applications of the q -Fourier transform F_q to nonlinear models of partial differential equations. First we verify that the theorems proved in Section 3 imply that F_q transfers a q -Gaussian into a q_1 -Gaussian if $q \geq 1$, $q_1 = (1 + q)/(3 - q)$. Moreover, as shown in [10], the operator $F_q : G_q \rightarrow G_{q_1}$ for $q > 1$ is invertible. These two facts have been essentially used in [7, 8, 9] for the proof of q -versions of the central limit theorem. Another application of F_q , as sketched below, shows that it can be used for establishing a relation between the porous medium equation and a nonlinear ordinary differential equation (ODE) similar to the usual Fourier transform.

The classic Fourier transform reduces the Cauchy problem for linear partial differential equations of the form $u_t(t, x) = A(D_x)u(t, x)$ $t > 0$, $x \in R^n$, $u(0, x) = \varphi(x)$, where $D_x = (D_1, \dots, D_n)$, $D_j = -i \frac{\partial}{\partial x_j}$, $j = 1, \dots, n$, and $A(D_x)$ is an elliptic differential operator, to an associated linear ODE with parameter $\xi \in R^n$. In the particular case of $n = 1$ and $A(D_x) = \frac{d^2}{dx^2}$ for the Fourier image $\hat{u}(t, \xi)$ of a solution $u(t, x)$, we have a dual differential equation

$$\hat{u}'_t(t, \xi) = -\xi^2 \hat{u}(t, \xi), \quad \hat{u}(0, \xi) = \hat{\varphi}(\xi), \quad (31)$$

where $\xi \in R^1$ is a parameter. This case corresponds to the Fokker-Planck equation for a Brownian motion without drift [14].

We now demonstrate the similar role of F_q in a simple model case, corresponding to the celebrated *porous medium equation* in the superdiffusion regime ubiquitously found in physical phenomena [16, 17, 18, 19, 20] (and references therein)¹. Consider the following non-linear diffusion equation with a singular diffusion coefficient,

$$\frac{\partial U}{\partial t} = (U^{1-q} U_x)_x, \quad t > 0, \quad x \in R^1, \quad q > 1. \quad (32)$$

We look for a solution in the similarity set $G_q^* = \{U(t, x) : U(t, x) = t^a G_q(\beta; t^b x), a = a(q), b = b(q) \in R^1, \beta = \beta(q) > 0\}$, where a and β do not depend on t and x .

Proposition 4.1 *Suppose $U(t, x) \in G_q^*$ is a solution to Eq. (32). Then its q -Fourier transform $\hat{U}_q(t, \xi) = F_q[U(t, x)](\xi)$ satisfies the following nonlinear*

¹The monograph [20] contains different approaches to the solution of the porous medium equation.

ordinary differential equation with parameter ξ

$$(\hat{U}_q)'_t = -\frac{B(\beta, q)\xi^2}{t^{\frac{q-1}{3-q}}}(\hat{U}_q)^{q_1}, \quad t > 0, \quad (33)$$

where $B(\beta, q) = \frac{2-q}{4\beta^{2-q}C_q^{q-1}}$ and $q_1 = \frac{1+q}{3-q}$.

Proof. Let $U \in G_q^*$ be a solution to (32), i.e. for some $a = a(q)$ and $\beta = \beta(q)$ it has representation $U(t, x) = t^a G_q(\beta; t^a x)$. Then, it follows from Proposition 2.1 that,

$$\begin{aligned} \hat{U}_q(t, \xi) &= F_q[U(t, x)](\xi) \\ &= F_q[G_q(\beta; x)]\left(\frac{\xi}{t^{a(2-q)}}\right) = \frac{1}{C_q} Y\left(q, \left(\frac{\sqrt{\beta}}{C_q}\right)^{q-1} \frac{\xi}{\sqrt{\beta} t^{a(2-q)}}\right), \end{aligned}$$

where $Y(q, \xi)$ is a solution to equation (20). Computing the derivative of $\hat{U}_q(t, x)$ in variable t , taking into account that $a = -1/(3-q)$ (see, e.g. [20]), and using equation (20), we obtain

$$(\hat{U}_q)_t = -\frac{2-q}{4\beta^{2-q}C_q^{2(q-1)}}\xi^2(\hat{U}_q)^{q_1},$$

where $q_1 = (1+q)/(3-q)$. ■

The inverse statement, given in the following formulation, is also true.

Proposition 4.2 *Suppose $V(t, \xi)$, $V(0, \xi) = 1$, is a solution to ODE with parameter ξ*

$$V' = -\frac{B(\beta, q)\xi^2}{t^{\frac{q-1}{3-q}}}V^{q_1}, \quad t > 0, \quad (34)$$

where $B(q, \beta)$ and q_1 are as in Proposition 4.1. Then its inverse q -Fourier transform $U(t, x) = F_q^{-1}[V(t, \xi)](x)$ exists and satisfies equation (32).

Proof. By separation of variables of (34) one can verify that its solution

$$V(t, \xi) = e_{q_1}^{-\frac{3-q}{8\beta^{2-q}C_q^{q-1}}\left(\xi t^{\frac{2-q}{3-q}}\right)}.$$

By Theorem 0.6 of paper [10] the inverse q -Fourier transform for $V(t, \xi)$ exists, and by virtue of Propositions 2.1 and 2.4 it has the representation

$$U(t, x) = \frac{1}{t^{\frac{1}{3-q}}} G_q\left(\beta(q); \frac{x}{t^{\frac{1}{3-q}}}\right), \quad \text{where } \beta(q) = \frac{1}{\left[2(3-q)C_q^{\frac{1}{q-1}}\right]^{\frac{2}{3-q}}}. \quad (35)$$

The latter is a solution to (32); see [20]. ■

Notice that, if the initial condition is given in the form $U(0, x) = \delta(x)$ with the Dirac's delta, and $q = 1$, then we obtain equation (31) ($\hat{\varphi}(\xi) \equiv 1$), in which $\beta = 1/4$, $B(\beta, 1) = 4\beta = 1$.

In order to study price fluctuations in stock markets it was introduced in [15] a stochastic process X_t defined from a stochastic differential equation $dX_t = \tau X_t + \sigma d\Omega_t$, where τ and σ are the drift and volatility coefficients respectively, and Ω_t is a solution to the Ito stochastic differential equation

$$d\Omega_t = [P(\Omega_t)]^{\frac{1-q}{2}} dB_t. \quad (36)$$

In this equation B_t is a Brownian motion, and P is a q -Gaussian distribution function. The corresponding Fokker-Planck type equation reads

$$\frac{\partial V(x, t|x', t')}{\partial t} = ([V(x, t|x', t')]^{2-q})_{xx},$$

which can easily be reduced to the form (32). From the financial applications point of view it is important to know the properties of the stochastic process X_t , since it can be considered as a q -alternative to the Brownian motion. One can effortlessly verify that if $U(t, x)$ is a solution to equation (32) for $t > 0$ with an initial condition $U(0, x) = f(x)$, then a solution $V(t, x)$, $t > t'$ to the same equation (32) considered for $t > t'$ with an initial condition $V(t', x) = f(x)$ can be represented in the form $V(t, x) = U(t - t', x)$, $t > t'$. It follows that X_t has stationary increments.

Concluding the discussion we note that equation (35) corresponds to the solution obtained from an ansatz [19] which has been at the base of the generalised Central Limit Theorem presented in [7].

5 Conclusion

Summarising, we have the following general picture for the q -Fourier transform of q -Gaussians.

1. The case $1 \leq q < 3$: for these values of q

(1a) the q -Fourier transform acts as $F_q : \mathcal{G}_q \rightarrow \mathcal{G}_{q'}$;

(1b) the relation between q and q' is given by $q' = \frac{1+q}{3-q}$.

2. The cases $q = \frac{1}{2}$ or $q = \frac{2}{3}$: for these two values of q the operator acts as $F_q : \mathcal{G}_q \rightarrow \mathcal{G}_0$, however the relationship (1b) is failed.

3. The case $q < 1$, but $q \neq \frac{1}{2}, \frac{2}{3}$: in this case (1a) is failed as well in the sense that there is no q' such that the q -Fourier transform of a q -Gaussian would be a q' -Gaussian.

The lesson we have learnt from the above analysis is that the operator F_q defined by formula (1) (or, the same, by formula (3)) is rich in content and applicable only if $q \in [1, 3)$. Its important application is given in [7] in the prove of the q -central limit theorem and in [11] in conjunction with (q, α) -stable distributions. Another application of F_q to the porous medium equation and related stochastic differential models with time dependent variance are discussed in Section 4 of the current paper. What concerns the case $q < 1$, the q -Fourier transform defined by formula (1) is not meaningful. An appropriate alternative definition of F_q in this case is remaining a challenging open question.

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